Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman and Kalman problems to hidden chaotic attractor in Chua circuits

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Content

- Computation of self-excited attractors and hidden attractors (hidden periodic oscillations and hidden chaotic attractors)
- Hidden attractors in applied models
  - Phase-locked loop (PLL)
  - Aircrafts control systems (windup and antiwindup)
  - Drilling systems and electrical machines
  - Secure (chaotic) communications
- Analytical-numerical methods for hidden attractor localization
  - 16th Hilbert problem on limit cycles
  - Aizerman conjecture and Kalman conjecture on absolute stability of control systems
  - Hidden chaotic attractor in Chua system
Computation of oscillations and attractors

**self-excited attractor localization:** *standard computational procedure* is 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches an self-excited oscillation and localizes it.

Van der Pol
\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \varepsilon(1-x^2)y
\end{align*} \]

Lorenz
\[ \begin{align*}
x &= -\sigma(x - y) \\
y &= rx - y - xz \\
z &= -bz + xy
\end{align*} \]

**hidden attractor:** *if basin of attraction does not intersect with a small neighborhood of equilibria* [Leonov, Kuznetsov, Vagaitsev, *Phys. Lett. A*, 2011]

✓ *standard computational procedure* does not work: all equilibria are stable or not in the basin of attraction
✓ integration with random initial data does not work: basin of attraction is small, system’s dimension is large

How to choose initial data in the attraction domain?
Hidden oscillation in Phase-locked loop (PLL)

PLL in microprocessor i486DX2-50 (1992, Ian Young) in Turbo regime stable operation is not guaranteed

N.A. Gubar’ (1961)
a simple PLL model:

\[
\dot{\eta} = \alpha \eta - (1 - a\alpha)(\text{sign} \sin(\sigma) - \gamma) \\
\dot{\sigma} = \eta - a(\text{sign} \sin(\sigma) - \gamma)
\]

hidden oscillation:

numerically any trajectory here tends to an equilibrium

physically only bounded attraction domain

Hidden oscillation in aircraft control systems

**Windup** – oscillations with increasing amplitude
- Crash - YF-22 Raptor (Boeing) 1992
- Crash - JAS-39 Gripen (SAAB) 1993

**Antiwindup** – an additional scheme to avoid windup effect in system with saturation

Lauvdal, Murray, Fossen, Stabilization of integrator chains in the presence of magnitude and rate saturations; a gain scheduling approach, *Proc. of CDC, 1997*: “Since stability in simulations does not imply stability of the physical control system (an example is the crash of the YF22) stronger theoretical understanding is required”

Hidden oscillation: Drilling system

Drill string failure \( \approx 1 \) out of 7 drilling rigs, costs $100 000 (www.oilgasprod.com)

De Bruin, J.C.A. et al. (2009), *Automatica*, 45(2), 405–415

\[
\begin{align*}
\dot{\omega}_u &= -k_\theta \alpha - T_{fu}(\omega_u) + k_m u, \\
\dot{\omega}_l &= k_\theta \alpha - T_{fl}(\omega_l), \\
\dot{\alpha} &= \omega_u - \omega_l, \\
\omega_{u,l} &= \dot{\theta}_{u,l}, \quad \alpha = \theta_u - \theta_l, \\
T_{fu}, T_{fl} &- friction torque
\end{align*}
\]

Operating mode: angular displacement between upper and lower discs \( \alpha = \theta_u - \theta_l \rightarrow \text{const.} \)

Hidden oscillation: stable limit cycle coexists with stable equilibrium

Hidden oscillation of \( (\theta_u - \theta_l) \):
- is difficult to find by standard simulation
- may lead to breakdown
Hidden oscillation: Drilling system with induction motor

Hidden oscillation: stable limit cycle coexists with stable equilibrium

Hidden oscillation of \((\theta_u - \theta_l)\):
— is difficult to find by standard simulation;
— may lead to breakdown

\[
\begin{align*}
\dot{y} &= -cy - s - xs, \\
\dot{x} &= -cx + ys, \\
\dot{\theta} &= u - s, \\
\dot{s} &= \frac{k_\theta}{J_u} \theta + \frac{b}{J_u}(u - s) + \frac{a}{J_u} y, \\
\dot{u} &= -\frac{k_\theta}{J_l} - \frac{b}{J_l}(u - s) + \frac{1}{J_l} T_{fl}(\omega - u),
\end{align*}
\]

Hidden oscillation in secure communication


synchronization is for self-excited, not for hidden Chua attractors


G.A.Leonov, N.V.Kuznetsov Hidden attractors in dynamical systems 5th IWCFTA, China (2012) 8/30
Computation of *self-excited attractors* and *hidden attractors* (hidden periodic oscillations and hidden chaotic attractors)

- Hidden attractors in applied models
  - Phase-locked loop (PLL)
  - Aircrafts control systems (windup and antiwindup)
  - Drilling systems and electrical machines
  - Secure (chaotic) communications

- Analytical-numerical methods for hidden attractor localization
  - 16th Hilbert problem on limit cycles
  - Aizerman conjecture and Kalman conjecture on absolute stability of control systems
  - Hidden chaotic attractor in Chua system
Hidden oscillations (2d): nested limit cycles

1900: 16th Hilbert problem (second part)
Number and mutual disposition of limit cycles

\[
\begin{align*}
\dot{x} &= P_n(x, y) = a_1 x^2 + b_1 xy + c_1 y^2 + \alpha_1 x + \beta_1 y + \ldots \\
\dot{y} &= Q_n(x, y) = a_2 x^2 + b_2 xy + c_2 y^2 + \alpha_2 x + \beta_2 y + \ldots
\end{align*}
\]

Problem is not solved even for quadratic systems (QS):

- N.N. Bautin 1949-1952: 3 limit cycles (LCs) [around one focus]
- I.G. Petrovskii, E.M. Landis 1955–1959: only 3 LCs
- L. Chen & M. Wang, S. Shi 1979-80: 4 LCs [(1,3), 2 focuses]
- R. Bamon 1985: number of LCs in QS is finite
- P. Zhang 2001: two focuses \(\Rightarrow\) only (1,n) distribution

Number of limit cycles \(H(n)\): \(H(2) \geq 4\)

Numerical methods: nested cycles are hidden oscillations
Computation (visualization) of limit cycles

- small-amplitude limit cycles: only analytical methods
  Lyapunov values: weak focus & Andronov-Hopf bifurcation

- normal-amplitude limit cycles: analytical & numerical methods
  
  - A. Kolmogorov: Calculation of limit cycles in two-dimensional quadratic systems
  
  - V. Arnold: Estimation of parameters domain corresponding to existence of limit cycles

V. Arnold wrote (2005): *To estimate the number of LCs of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech. & Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC!*... The fact that this did not occur suggests that the above-mentioned domains are, apparently, small.
Direct method for computation of Lyapunov values in Euclidian coordinates and in the time domain

\[ \dot{x} = -y + f(x, y) = -y + \sum_{k+j=2}^{n} f_{k,j} x^k y^j + o\left((|x| + |y|)^n\right), \quad x(t, h) = x(t, 0, h) \]

\[ \dot{y} = +x + g(x, y) = +x + \sum_{k+j=2}^{n} g_{k,j} x^k y^j + o\left((|x| + |y|)^n\right), \quad y(t, h) = y(t, 0, h) \]

1. Approximation of solution \( x(t, h), y(t, h) \)

\[ x(t, h) = \sum_{k=1}^{n} \tilde{x}_{h^k}(t) h^k + o(h^n), \quad y(t, h) = \sum_{k=1}^{n} \tilde{y}_{h^k}(t) h^k + o(h^n) \]

2. Approximation of return time \( T(h) : x(T(h), h) = 0 \)

\[ T(h) = 2\pi + \Delta T(h) = 2\pi + \sum_{j=1}^{n} \tilde{T}_j h^j + o(h^n) \]

3. Computation of Lyapunov values \( L_k : \{\tilde{L}_{ij}\}_{i=2}^{2k} = 0 \)

\[ y(T(h), h) = h + \tilde{L}_2 h^2 + \tilde{L}_3 h^3 + \tilde{L}_4 h^4 + \ldots + o(h^n) = h + L_k h^{2k+1} + o(h^{2k+1}) \]

In the study of real systems in applied problems it is more convenient to study the system in the initial "physical" space.

Lyapunov values: in terms of system’s coefficients

To compute general expression of $k$th Lyapunov value it is necessary to consider expansion upto $2k + 1$: $L_k = L_k(\{g_{k,j}\}_{k+j=2}^{2k+1}, \{f_{k,j}\}_{k+j=2}^{2k+1})$

\[
\dot{x} = -y + f_2 x^2 + f_{11} xy + f_{02} y^2 + \ldots, \quad \dot{y} = x + g_2 x^2 + g_{11} xy + g_{02} y^2 + \ldots
\]

- **1949**, Bautin:
  
  $L_1 = \frac{\pi}{4} (g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_2 f_{11} + f_{02} f_{11} - g_{11} g_2 + 2g_{02} f_{02} - 2f_2 g_{20} - g_0 g_{11})$

- **1959**, Serebryakova: $L_2 = \frac{\pi}{72}$

- **1968**, Shuko: first computer program for Lq calculation

- **2008**, Kuznetsov, Leonov: $L_3 = \frac{\pi}{1728}$

To simplify LV expressions, it is often used change of coordinates (complex, polar) & reduction to normal forms (but such reductions is not unique and often laborious).

Lyapunov values & small limit cycles:
Andronov-Hopf bifurcation, cyclicity and center problems

\[
\begin{align*}
\dot{x} &= f_{10}x + f_{01}y + f(x, y), \\
\dot{y} &= g_{10}x + g_{01}y + g(x, y)
\end{align*}
\]

Solution \( x(t, h) = x(t, 0, h), \ y(t, h) = y(t, 0, h), \) return time \( T(h) \)

Small limit cycles: \( L_0 = \tilde{L}_1 = 0, \ L_1 = \tilde{L}_3 > 0 \)

\[
y(T(h), h) - h = L_1 h^3 + o(h^3)
\]

\[
g_{01}^\varepsilon = g_{01} + \varepsilon_1, \ g_{03}^\varepsilon = g_{03} + \varepsilon_3
\]

\[
L_0^\varepsilon = \tilde{L}_1^\varepsilon < 0 < L_1^\varepsilon = \tilde{L}_3^\varepsilon, \ |L_0^\varepsilon| << |L_1^\varepsilon|
\]

\[
y(T(h), h) - h = \tilde{L}_1^\varepsilon h + \tilde{L}_2^\varepsilon h^2 + \tilde{L}_3^\varepsilon h^3 + o(h^3): \quad \exists h_1, h_2: y(T(h_1), h_1) - h_1 < 0 < y(T(h_2), h_2) - h_2
\]

Number of "independent" zeros of Lyapunov values expressions?
Algebraic methods for analysis of polynomials:
Bautin ideal, Groebner basis ...

- \( C(2) = 3, \) Bautin 1949
- \( C(3) \geq 11, \) Zoladek 1995
- \( C(n) = ?, \)
  e.g., a lower bound of LC
  Lynch 2005; Han&Li 2012
Four limit cycles in quadratic system

Small limit cycles:

\[ L_0 = 0, \quad L_1 > 0, \quad \tilde{L}_0 < 0 < \tilde{L}_1, |\tilde{L}_0| << |\tilde{L}_1| \]

\[ y(T(h), h) - h = L_0 h + L_1 h^3 + o(h^4) \]

\[ L_1 = \frac{-\pi}{4(-\alpha^2)^{5/2}}(\alpha_2(b_2c_2 - 1) - a_2(b_2 + 2)) \]

\[ L_2 = \frac{\pi(b_2-3)(b_2 c_2-1)^{5/2}}{24(-a_2)^{7/2}(2+b_2)^{7/2}}((c_2 b_2 + b_2 - 2c_2)(c_2 b_2 - 1) - a_2(c_2 - 1)(1+2c_2)^2) \]

\[ L_3 = \frac{\pi\sqrt{5}(3c_2-1)^{9/2}}{500000(-a_2)^{9/2}}(c_2 - 2)(4c_2^3 a_2 - 3c_2^2 - 3a_2 c_2 - 8c_2 - a_2 + 3) \]

**Theorem:** Quadratic system has 4 limit cycles, if

1/3 < \( c_2 < 1, \quad 1 < b_2 < 3, \quad 4a_2(c_2 - 1) > (b_2 - 1)^2, b_2 c_2 > 1, \]

0 < \( \beta_2 < \varepsilon, \quad \alpha_2 \in \left( \frac{a_2(2 + b_2)}{b_2 c_2 - 1}, \frac{a_2(2 + b_2)}{b_2 c_2 - 1} + \delta \right), \quad 1 \gg \delta \gg \varepsilon \geq 0. \]

Visualization of 4 normal size limit cycles in QS

\[ \begin{align*}
\dot{x} &= x^2 + xy + y, & \dot{y} &= ax^2 + bxy + cy^2 + \alpha x + \beta y \\
c &\in (1/3, 1), & \alpha &= -\varepsilon^{-1}, & bc &< 1, & b &> a + c, & 2c &< b + 1, & 4a(c-1) &> (b-1)^2, & \beta & = 0
\end{align*} \]

**Theorem.** For sufficiently small \( \varepsilon \) the system has three limit cycles: one to the left of line \( \{ x = -1 \} \) and two to the right of it.

(Increase \( \beta \) and get four normal size limit cycles)

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Kuznetsov, Kuznetsova, Leonov, Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, Diff. eq. and Dyn. syst., 2012 (doi: 10.1007/s12591-012-0118-6)
Hidden oscillations (3d): Aizerman and Kalman conjectures

If $\dot{z} = Az + bk^*z$, is asympt. stable $\forall k \in (k_1, k_2) : \forall z(t, z_0) \to 0$, then is $\dot{x} = Ax + b\varphi(\sigma), \sigma = c^*x, \varphi(0) = 0, k_1 < \varphi(\sigma)/\sigma < k_2 : \forall x(t, x_0) \to 0$?

1949: $k_1 < \varphi(\sigma)/\sigma < k_2$

1957: $k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true (Aizerman $- n \geq 2$, Kalman $- n \geq 4$). Periodic solution can exist for nonlinearity from linear stability sector.

Kalman problem (Kalman conjecture) 1957

\[ \dot{x} = Ax + b\varphi(\sigma), \quad \sigma = c^*x, \quad \varphi(0) = 0, \quad 0 < \varphi'(\sigma) < k : \quad \forall x(t, x_0) \to 0? \]

- **Fitts R. 1966**: series of counterexamples in \( \mathbb{R}^4 \), nonlinearity \( \varphi(\sigma) = \sigma^3 \) (some of them were reported being false, but some — true)

- **Barabanov N. 1979-1988**: proved Kalman conj. is true for \( \mathbb{R}^3 \); analytical ‘counterex.’ construction in \( \mathbb{R}^4 \), \( \varphi(\sigma) ‘close’ \) to \( \text{sign}(\sigma) (d/d\sigma \not< 0) \) later ‘gaps’ were reported by Glutsyuk, Meisters, Bernat & Llibre

- **Leonov G. 1996**: proved Kalman conj. is true in \( \mathbb{R}^3 \) (by freq. methods)

- **Bernat J. & Llibre J. 1996**: analytical-numerical ‘counterex.’ construction in \( \mathbb{R}^4 \), \( \varphi(\sigma) ‘close’ \) to \( \text{sat}(\sigma) (d/d\sigma \not< 0) \)

- **Leonov G., Kuznetsov N., Bragin V. 2010**: analytical-numerical counterexamples construction for any type of nonl.; counterexample in \( \mathbb{R}^4 \) with \( \varphi(\sigma) = \tanh(\sigma) \): \( 0 < \tanh'(\sigma) \leq 1 \)

Describing function method (DFM) can lead to untrue results: no periodic solution for Aizerman’s or Kalman’s conditions by DFM

\[
\begin{align*}
(1) \quad \dot{x} &= P_0 x + \varphi(x) \\
(2) \quad \dot{x} &= P_0 x + \varepsilon \varphi(x) \\
(3) \quad \dot{x}_j &= P_0 x_j + \varepsilon_j \varphi(x_j)
\end{align*}
\]

\(\varepsilon\) allows one to justify math. strictly DFM for (2) & to determine a stable nontrivial periodic solution \(x^0(t) \rightarrow \text{oscillating attractor } A_0\).

Localization of attractor \(A\) in (1): numerically follow transformation of \(A_j\) with increasing \(j = 0, \ldots, m\) (\(A_m = A\)). Two cases are possible:

1. if all points of \(A_0\) are in the attraction domain of \(A_1\) (oscillating attractor of (3) with \(j = 1\)), then solution \(x^1(t)\) can be determined numerically by starting a trajectory of (3) with \(j = 1\) from initial point \(x^0(0)\). If in computational process \(x^1(t)\) is not fallen to equilibria and is not \(\rightarrow \infty\) (on suff. large \([0, T]\)), then \(x^1(t)\) computes attractor \(A_1\). Then perform similar procedure for (3) with \(j = 2\): by starting trajectory \(x^2(t)\) of (3) with \(j = 2\) from init. point \(x^1(T)\) (last point on previous step) we compute \(A_2\). And so on.

2. in the change from system (2) to (3) with \(j = 1\), it’s observed loss of stability bifurcation and vanishing of attractor \(A_0\) (or \(A_{j-1}\) on j-th step).
DFM justification for critical case

\[ \dot{x} = P x + q \varphi_\varepsilon (r^* x), \quad x \in \mathbb{R}^n \]
\[ \text{eigs}(P): \lambda_{1,2} = \pm i \omega_0, \quad \text{Re } \lambda_j > 2 < 0 \]
\[ \Rightarrow \exists x = S y : \]
\[ \dot{y}_1 = -\omega_0 y_1 + b_1 \varphi_\varepsilon (y_1 + c_3^* y_3) \]
\[ \dot{y}_2 = +\omega_0 y_1 + b_2 \varphi_\varepsilon (y_1 + c_3^* y_3) \]
\[ \dot{y}_3 = A_3 y_3 + b \varphi_\varepsilon (y_1 + c_3^* y_3) \]
\[ A_3 \text{-stable } (n-2) \times (n-2) \text{-matrix} \]
\[ b, c \quad (n-2) \text{-vectors}. \]

Theorem. If \( b_1 < 0 \) and \( 0 < (\mu b_2 \omega_0 (c_3^* b_3 + b_1) + b_1 \omega_0^2) \)
then for suff. small \( \varepsilon \) exists a stable periodic solution with the initial data
\[ y_1(t) = -\sin(\omega_0 t) y_2(0) + O(\varepsilon), \quad y_2(t) = \cos(\omega_0 t) y_2(0) + O(\varepsilon), \quad y_3(t) = O(\varepsilon) \]
\[ y_1(0) = O(\varepsilon^2), \quad y_2(0) = -\sqrt{\frac{\mu (\mu b_2 \omega_0 (c_3^* b_3 + b_1) + b_1 \omega_0^2)}{-3 \omega_0^2 M b_1}} + O(\varepsilon), \quad y_3(0) = O(\varepsilon^2) \]
Counterexample to Aizerman and Kalman conjecture

\[
\begin{align*}
\dot{x}_1 &= -x_2 - 10 \varphi(\sigma) \\
\dot{x}_2 &= x_1 - 10.1 \varphi(\sigma) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_3 - x_4 + \varphi(\sigma) \\
\sigma &= x_1 - 10.1 x_3 - 0.1 x_4
\end{align*}
\]

**Thm:** $\varphi(\sigma) = \varphi^0(\sigma)$ \exists periodic solution with $x_1(0) = x_3(0) = x_4(0) = 0, x_2(0) = -1.7513$

Aizerman’s conjecture: $0 \leq \varphi^j(\sigma) \leq 1$,

$$\varphi^j(\sigma) = \begin{cases} 
\sigma, & |\sigma| \leq \varepsilon_j; \\
\text{sign}(\sigma)\varepsilon_j^3, & |\sigma| > \varepsilon_j
\end{cases} \quad \varepsilon_j = 0.1, \ldots, 1, \quad \varphi^{10}(\sigma) = \text{sat}(\sigma)$$

Kalman’s conjecture: $iN \leq \psi^i(\sigma) \leq 1 \quad 0 < \frac{d}{d\sigma}\tanh(\sigma) \leq 1$

$$\psi^i(\sigma) = \begin{cases} 
\sigma, & |\sigma| \leq 1; \\
\text{sign}(\sigma) + i(\sigma - \text{sign}(\sigma))N, & |\sigma| > 1
\end{cases} \quad N = 0.01, i = 1, \ldots, 5$$

$$\theta^i(\sigma) = \text{sat}(\sigma) + i(\tanh(\sigma) - \text{sat}(\sigma))/10 \quad i = 1, \ldots, 10 \quad \theta^{10}(\sigma) = \tanh(\sigma)$$
Counterexample to Aizerman conjecture
Smooth counterexample to Kalman conjecture

\[ \dot{x}_1 = -x_2 - 10 \varphi(\sigma) \]
\[ \dot{x}_2 = x_1 - 10.1 \varphi(\sigma) \]
\[ \dot{x}_3 = x_4 \]
\[ \dot{x}_4 = -x_3 - x_4 + \varphi(\sigma) \]
\[ \sigma = x_1 - 10.1 x_3 - 0.1 x_4 \]

\[ \varphi(\sigma) = \theta^i(\sigma) = \text{sat}(\sigma)+i(\tanh(\sigma)-\text{sat}(\sigma))/10 \]
\[ i = 1, \ldots, 10 \]
\[ \tanh(\sigma) = \frac{e^\sigma-e^{-\sigma}}{e^\sigma+e^{-\sigma}} \]

Counterexample to Kalman problem (i=10)
(0 < \frac{d}{d\sigma}\tanh(\sigma) \leq 1)
periodic solution exists,
linear systems are stable
Attractors in Chua’s circuits

\[
\begin{align*}
\dot{x} &= \alpha(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= - (\beta y + \gamma z),
\end{align*}
\]

\[
f(x) = m_1 x + \text{sat}(x) = m_1 x + \frac{1}{2} (m_0 - m_1) (|x + 1| + |x - 1|)
\]

Chua circuit can be used in chaotic communications


Could an attractor exist and how to localize it, if equilibrium is stable?

L. Chua, 1992: If zero equilibrium is stable \(\Rightarrow\) there is no attractor
Hidden attractor in classical Chua’s system

In 2010 the notion of *hidden attractor* was introduced and hidden chaotic attractor was found for the first time by the authors.

\[
\dot{x} = \alpha(y - x - m_1 x - \psi(x))
\]
\[
\dot{y} = x - y + z,\quad \dot{z} = -(\beta y + \gamma z)
\]
\[
\psi(x) = (m_0 - m_1)\text{sat}(x)
\]
\[
\alpha = 8.4562, \beta = 12.0732, \gamma = 0.0052
\]
\[
m_0 = -0.1768, m_1 = -1.1468
\]

Stable zero eqv. and 2 symmetric saddles: trajectories "from" saddles tend to zero eqv. or to infinity: black and red

Hidden attractor(green)
Transition to chaos scenario

\[ \varepsilon = 0.1 \]

\[ \varepsilon = 0.2 \]

\[ \varepsilon = 0.3 \]

\[ \varepsilon = 0.5 \]

\[ \varepsilon = 0.7 \]

\[ \varepsilon = 0.9 \]
References: 2012

References: 2011


Questions and remarks