Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman and Kalman problems to hidden chaotic attractor in Chua circuits

Gennady Leonov, Nikolay Kuznetsov

Department of Applied Cybernetics
Faculty of Mathematics and Mechanics
Saint-Petersburg State University
leonov@math.spbu.ru, kuznetsov@math.spbu.ru

Content

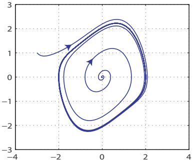
- Computation of *self-excited attractors* and *hidden attractors* (hidden periodic oscillations and hidden chaotic attractors)
- Hidden attractors in applied models
  - Phase-locked loop (PLL)
  - Aircrafts control systems (windup and antiwindup)
  - Drilling systems and electrical machines
  - Secure (chaotic) communications
- Analytical-numerical methods for hidden attractor localization
  - 16th Hilbert problem on limit cycles
  - Aizerman conjecture and Kalman conjecture on absolute stability of control systems
  - Hidden chaotic attractor in Chua system

Computation of oscillations and attractors

self-excited attractor localization: *standard computational procedure* is 1) to find equilibria; 2) after transient process trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, reaches an self-excited oscillation and localizes it.

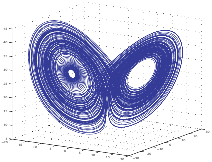
Van der Pol

$$\dot{x} = y$$
$$\dot{y} = -x + \varepsilon(1-x^2)y$$



Lorenz

$$x = -\sigma(x - y)$$
$$y = rx - y - xz$$
$$z = -bz + xy$$

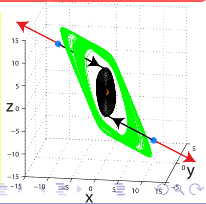


hidden attractor: *if basin of attraction does not intersect with a small neighborhood of equilibria* [Leonov, Kuznetsov, Vagaitsev, *Phys.Lett.A*, 2011]

✓ *standard computational procedure* does not work: all equilibria are stable or not in the basin of attraction

✓ integration with random initial data does not work: basin of attraction is small, system's dimension is large

How to choose initial data in the attraction domain?

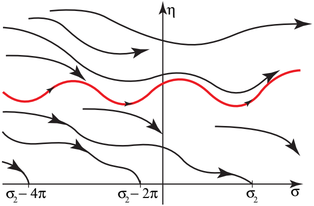


Hidden oscillation in Phase-locked loop (PLL)

PLL in microprocessor i486DX2-50 (1992, Ian Young)
in Turbo regime stable operation is not guaranteed

N.A. Gubar' (1961)
a simple PLL model:

$$\dot{\eta} = \alpha\eta - (1 - a\alpha)(\operatorname{sign}\sin(\sigma) - \gamma)$$
$$\dot{\sigma} = \eta - a(\operatorname{sign}\sin(\sigma) - \gamma)$$



hidden oscillation:

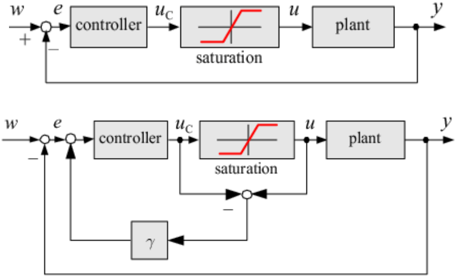
numerically any trajectory here tends to an equilibrium
physically only bounded attraction domain

Leonov G.A., Kuznetsov N.V., Yuldahsev M.V., Yuldashev R.V., Analytical method for computation of phase-detector characteristic, IEEE Transactions on Circuits and Systems Part II, vol. 59, num. 10, 2012 (doi: 10.1109/TCSII.2012.2213362)

Windup – oscillations with increasing amplitude

- Crash - YF-22 Raptor (Boeing) 1992
- Crash - JAS-39 Gripen (SAAB) 1993

Antiwindup – an additional scheme to avoid windup effect in system with saturation



Lauvdal, Murray, Fossen, Stabilization of integrator chains in the presence of magnitude and rate saturations; a gain scheduling approach, *Proc. of CDC, 1997*: *"Since stability in simulations does not imply stability of the physical control system (an example is the crash of the YF22) stronger theoretical understanding is required"*

Survey: G.A. Leonov, B.R. Andrievsky, A.Yu. Pogromsky, Control of aircrafts with AW-compensation, Differential equations, N13, 2012

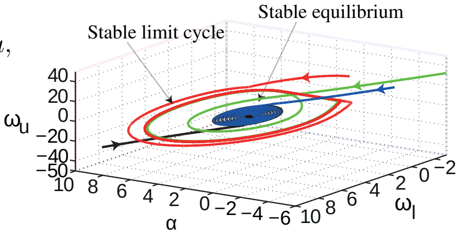
Hidden oscillation: Drilling system

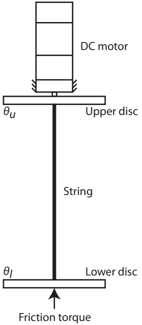
Drill string failure ≈ 1 out of 7 drilling rigs, costs \$100 000 (www.oilgasprod.com)

De Bruin, J.C.A. et al. (2009), *Automatica*, **45**(2), 405–415

$$\dot{\omega}_u = -k_\theta \alpha - T_{fu}(\omega_u) + k_m u,$$
$$\dot{\omega}_l = k_\theta \alpha - T_{fl}(\omega_l),$$
$$\dot{\alpha} = \omega_u - \omega_l,$$
$$\omega_{u,l} = \dot{\theta}_{u,l}, \ \alpha = \theta_u - \theta_l,$$

T_{fu}, T_{fl} — friction torque





Operating mode: angular displacement between upper and lower discs $\alpha = \theta_u - \theta_l \to$ const.

Hidden oscillation: stable limit cycle coexists with stable equilibrium

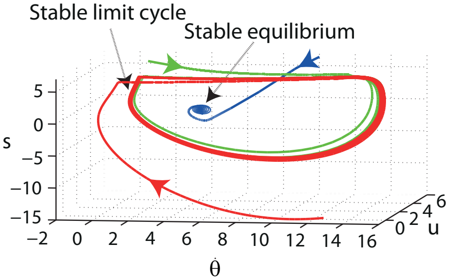
Hidden oscillation of $(\theta_u - \theta_l)$:
– is difficult to find by standard simulation
– may lead to breakdown

Hidden oscillation: Drilling system with induction motor

Hidden oscillation: stable limit cycle coexists with stable equilibrium

Hidden oscillation of $(\theta_u - \theta_l)$:
— is difficult to find by standard simulation;
— may lead to breakdown

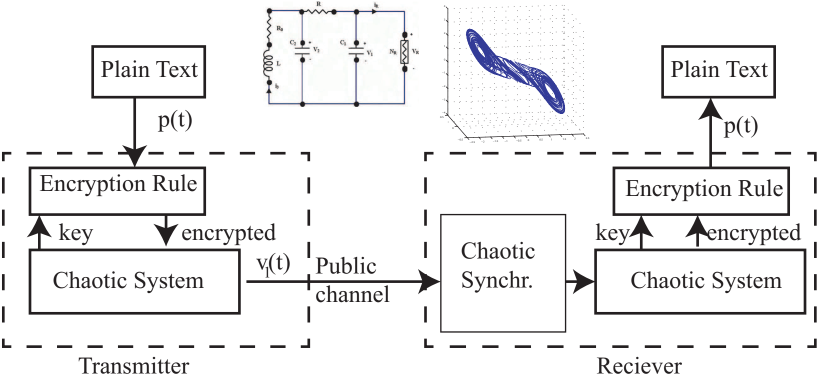
$$\dot{y} = -cy - s - xs, \dot{x} = -cx + ys,$$
$$\dot{\theta} = u - s,\ \ s = -\dot{\theta}_u,\ \ u = -\dot{\theta}_l$$
$$\dot{s} = \frac{k_\theta}{J_u}\theta + \frac{b}{J_u}(u - s) + \frac{a}{J_u}y,$$
$$\dot{u} = -\frac{k_\theta}{J_l} - \frac{b}{J_l}(u - s) + \frac{1}{J_l}T_{fl}(\omega - u),$$



Kiseleva, Kuznetsov, Leonov, Neittaanmaki, Drilling Systems Failures and Hidden Oscillations, NSC 2012 - 4th IEEE International Conference on Nonlinear Science and Complexity, 2012, 109-112 (doi:10.1109/NSC.2012.6304736)

Hidden oscillation in secure communication

Tao Yang, A Survey of Chaotic Secure Communication Systems, Int. J.of Computational Cognition, 2(2), 2004



synchronization is for self-excited, not for hidden Chua attractors

Leonov G.A., Kuznetsov N.V., Vagaitsev V.I., Hidden attractor in smooth Chua systems, Physica D, 241(18), 2012, 1482-1486 (doi: 10.1016/j.physd.2012.05.016)

Content

- Computation of *self-excited attractors* and *hidden attractors* (hidden periodic oscillations and hidden chaotic attractors)
- Hidden attractors in applied models
  - Phase-locked loop (PLL)
  - Aircrafts control systems (windup and antiwindup)
  - Drilling systems and electrical machines
  - Secure (chaotic) communications
- Analytical-numerical methods for hidden attractor localization
  - 16th Hilbert problem on limit cycles
  - Aizerman conjecture and Kalman conjecture on absolute stability of control systems
  - Hidden chaotic attractor in Chua system

Hidden oscillations (2d): nested limit cycles

1900: 16th Hilbert problem (second part)
Number and mutual disposition of limit cycles

$$\dot{x} = P_n(x, y) = a_1 x^2 + b_1 xy + c_1 y^2 + \alpha_1 x + \beta_1 y + \ldots$$
$$\dot{y} = Q_n(x, y) = a_2 x^2 + b_2 xy + c_2 y^2 + \alpha_2 x + \beta_2 y + \ldots$$

Problem is not solved even for quadratic systems (QS):

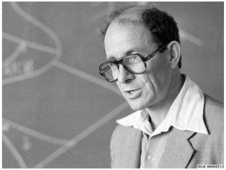
- N.N. Bautin 1949-1952: 3 limit cycles (LCs) [around one focus]
- I.G.Petrovskii, E.M.Landis 1955–1959: **only** 3 LCs
- L.Chen & M.Wang, S.Shi 1979-80: 4 LCs [(1,3), 2 focuses]
- R. Bamon 1985: number of LCs in QS is finite
- P. Zhang 2001: two focuses $\Rightarrow$ only (1,n) distribution

Number of limit cycles $H(n)$: $H(2) \geqslant 4$

Numerical methods: nested cycles are hidden oscillations

Computation (visualization) of limit cycles

- small-amplitude limit cycles: only analytical methods
  Lyapunov values: weak focus & Andronov-Hopf bifurcation
- normal-amplitude limit cycles: analytical&numerical methods

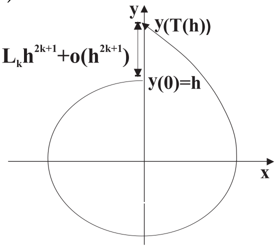


- A.Kolmogorov: Calculation of limit cycles in two-dimensional quadratic systems
- V.Arnold: Estimation of parameters domain corresponding to existence of limit cycles

V. Arnold wrote (2005): *To estimate the number of LCs of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech.&Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC!... The fact that this did not occur suggests that the above-mentioned domains are, apparently, small.*

Direct method for computation of Lyapunov values in Euclidian coordinates and in the time domain

$$\dot{x} = -y + f(x,y) = -y + \sum_{k+j=2}^{n} f_{kj} x^k y^j + o\big((|x|+|y|)^n\big), \quad x(t,h) = x(t,0,h)$$
$$\dot{y} = +x + g(x,y) = +x + \sum_{k+j=2}^{n} g_{kj} x^k y^j + o\big((|x|+|y|)^n\big), \quad y(t,h) = y(t,0,h)$$



1. Approximation of solution $x(t,h), y(t,h)$

$$x(t,h) = \sum_{k=1}^{n} \widetilde{x}_{h^k}(t) h^k + o(h^n), \quad y(t,h) = \sum_{k=1}^{n} \widetilde{y}_{h^k}(t) h^k + o(h^n)$$

2. Approximation of return time $T(h): x(T(h),h) = 0$

$$T(h) = 2\pi + \Delta T(h) = 2\pi + \sum_{j=1}^{n} \widetilde{T}_j h^j + o(h^n)$$

3. Computation of Lyapunov values $\mathrm{L_k}$: $\{\widetilde{\mathrm{L}}_{\mathrm{i}}\}_{\mathrm{i}=2}^{2\mathrm{k}} = 0$

$$y(T(h),h) = h + \widetilde{L}_2 h^2 + \widetilde{L}_3 h^3 + \widetilde{L}_4 h^4 + \ldots + o(h^n) \qquad = \mathbf{h} + \mathrm{L}_{\mathbf{k}} \mathbf{h}^{\mathbf{2k+1}} + \mathbf{o}(\mathbf{h}^{\mathbf{2k+1}})$$

In the study of real systems in applied problems it is more convenient to study the system in the initial "physical" space.

G.A. Leonov, N.V. Kuznetsov, and E.V. Kudryashova, A direct method for calculating Lyapunov quantities of two-dimensional dynamical systems, Proceedings of the Steklov Institute of Mathematics, 272(Suppl. 1), 2011, 119-127

Lyapunov values: in terms of system's coefficients

To compute general expression of kth Lyapunov value it is necessary to consider expansion upto $2k+1$: $\mathrm{L}_k = \mathrm{L}_k(\{g_{k,j}\}_{k+j=2}^{2k+1}, \{f_{k,j}\}_{k+j=2}^{2k+1})$

$$\dot{x} = -y + f_{20}x^2 + f_{11}xy + f_{02}y^2 + \ldots,\ \dot{y} = x + g_{20}x^2 + g_{11}xy + g_{02}y^2 + \ldots.$$

- **1949**, Bautin:

$$\mathrm{L}_1 = \frac{\pi}{4}(g_{21} + f_{12} + 3f_{30} + 3g_{03} + f_{20}f_{11} + f_{02}f_{11} - g_{11}g_{20} + 2g_{02}f_{02} - 2f_{20}g_{20} - g_{02}g_{11})$$

- **1959**, Serebryakova: $\mathrm{L}_2 = \frac{\pi}{72}($ ...1 page... $)$
- **1968**, Shuko: first computer program for Lq calculation
- **2008**, Kuznetsov, Leonov: $\mathrm{L}_3 = \frac{\pi}{1728}($ ...4 pages... $)$

To simplify LV expressions, it is often used change of coordinates (complex, polar) & reduction to normal forms (but such reductions is not unique and often laborious).

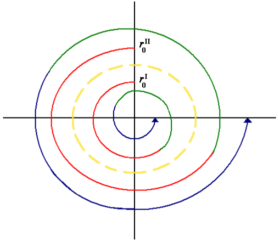
Survey: Leonov G.A., Kuznetsov N.V., Kudryashova E.V., Cycles of two-dimensional systems: computer calculations, proofs, and experiments, **Vestnik St. Petersburg University. Mathematics**, 41(3), **2008**, 216-250

Lyapunov values & small limit cycles:

Andronov-Hopf bifurcation, cyclicity and center problems

$\dot{x} = f_{10}x + f_{01}y + f(x,y), \quad \dot{y} = g_{10}x + g_{01}y + g(x,y)$

Solution $x(t,h) = x(t,0,h)$, $y(t,h) = y(t,0,h)$, return time $T(h)$



Small limit cycles: $\mathrm{L}_0 = \widetilde{L}_1 = 0, \mathrm{L}_1 = \widetilde{L}_3 > 0$

$y(T(h),h) - h = \mathrm{L}_1 h^3 + o(h^3)$

$g_{01}^{\varepsilon} = g_{01} + \varepsilon_1, \ g_{03}^{\varepsilon} = g_{03} + \varepsilon_3$

$\mathrm{L}_0^{\varepsilon} = \widetilde{L}_1^{\varepsilon} < 0 < \mathrm{L}_1^{\varepsilon} = \widetilde{L}_3^{\varepsilon}, \ |\mathrm{L}_0^{\varepsilon}| << |\mathrm{L}_1^{\varepsilon}|$

$y(T(h),h) - h = \widetilde{L}_1^{\varepsilon} h + \widetilde{L}_2^{\varepsilon} h^2 + \widetilde{L}_3^{\varepsilon} h^3 + o(h^3)$:

$\exists h_1, h_2 : y(T(h_1),h_1) - h_1 < 0 < y(T(h_2),h_2) - h_2$

Number of "independent" zeros of Lyapunov values expressions?
Algebraic methods for analysis of polynomials:
Bautin ideal, Groebner basis ...

- C(2)=3, Bautin 1949
- C(3)$\geq$ 11, Zoladek 1995
- C(n)=?,

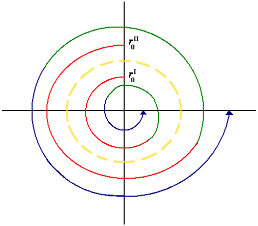
e.g., a lower bound of LC
Lynch 2005; Han&Li 2012

Four limit cycles in quadratic system

Small limit cycles:

$$\mathrm{L}_0 = 0, \quad \mathrm{L}_1 > 0, \quad \widetilde{\mathrm{L}}_0 < 0 < \widetilde{\mathrm{L}}_1, |\widetilde{\mathrm{L}}_0| << |\widetilde{\mathrm{L}}_1|$$

$$y(T(h), h) - h = \mathrm{L}_0 h + \mathrm{L}_1 h^3 + o(h^4)$$



$$L_1 = \frac{-\pi}{4(-\alpha_2)^{5/2}} (\alpha_2(b_2 c_2 - 1) - a_2(b_2 + 2)).$$

$$L_2 = \frac{\pi(b_2-3)(b_2\, c_2-1)^{5/2}}{24(-a_2)^{7/2}(2+b_2)^{7/2}} ((c_2 b_2 + b_2 - 2c_2)(c_2 b_2 - 1) - a_2(c_2 - 1)(1 + 2c_2)^2).$$

$$L_3 = \frac{\pi\sqrt{5}(3c_2-1)^{9/2}}{500000(-a_2)^{9/2}} (c_2 - 2)(4c_2^3 a_2 - 3c_2^2 - 3a_2 c_2 - 8c_2 - a_2 + 3).$$

Theorem: Quadratic system has 4 limit cycles, if

$$1/3 < c_2 < 1,\ 1 < b_2 < 3,\ 4a_2(c_2 - 1) > (b_2 - 1)^2,\ b_2 c_2 > 1,$$

$$0 < \beta_2 < \varepsilon,\ \alpha_2 \in \left(\frac{a_2(2 + b_2)}{b_2 c_2 - 1}, \frac{a_2(2 + b_2)}{b_2 c_2 - 1} + \delta\right),\ 1 \gg \delta \gg \varepsilon \geq 0.$$

Leonov G.A., Kuznetsova O.A., Lyapunov quantities and limit cycles of two-dimensional dynamical systems. Analytical methods and symbolic computation, Regular and chaotic dynamics, 15(2-3), 2010, 354-377

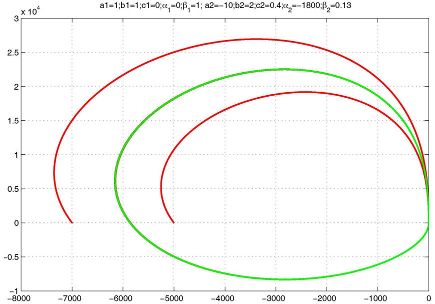
Visualization of 4 normal size limit cycles in QS

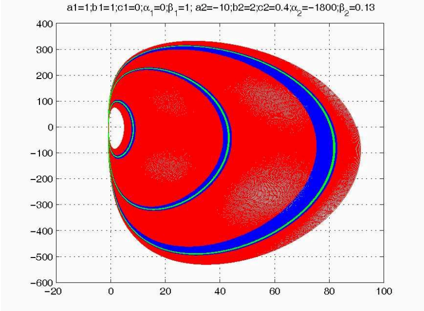
$$\dot{x} = x^2 + xy + y, \quad \dot{y} = ax^2 + bxy + cy^2 + \alpha x + \beta y$$

$$c \in (1/3, 1), \alpha = -\varepsilon^{-1}, bc < 1, b > a + c, 2c < b+1, 4a(c-1) > (b-1)^2, \beta = 0$$

Theorem. *For sufficiently small ε the system has three limit cycles: one to the left of line$\{x = -1\}$ and two to the right of it.*

(Increase β and get four normal size limit cycles)

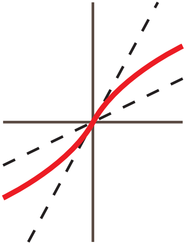




Kuznetsov, Kuznetsova, Leonov, Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, Diff. eq. and Dyn. syst., 2012 (doi: 10.1007/s12591-012-0118-6)

Hidden oscillations (3d): Aizerman and Kalman conjectures

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$, then is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}$, $\varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma < k_2$: $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \; k_1 < \varphi(\sigma)/\sigma < k_2$

$1957: \; k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true (Aizerman - $n \geq 2$, Kalman - $n \geq 4$). Periodic solution can exist for nonlinearity from linear stability sector.

Survey: Bragin, Vagaitsev, Kuznetsov, Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544*

Kalman problem (Kalman conjecture) 1957

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma),\ \sigma = \mathbf{c}^*\mathbf{x},\ \varphi(0) = 0,\ \ 0 < \varphi'(\sigma) < k:\ \ \forall \mathbf{x}(t, \mathbf{x}_0) \to 0?$

- **Fitts R. 1966**: series of counterexamples in $\mathbb{R}^4$, nonlinearity $\varphi(\sigma) = \sigma^3$ (some of them were reported being false, but some — true)
- **Barabanov N. 1979-1988**: proved Kalman conj. is true for $\mathbb{R}^3$; analytical 'counterex.' construction in $\mathbb{R}^4$, $\varphi(\sigma)$ 'close' to $\operatorname{sign}(\sigma)$ $\big(d/d\sigma \not> 0\big)$
  later 'gaps' were reported by Glutsyuk, Meisters, Bernat & Llibre
- **Leonov G. 1996**: proved Kalman conj. is true in $\mathbb{R}^3$ (by freq. methods)
- **Bernat J. & Llibre J. 1996**: analytical-numerical 'counterex.' construction in $\mathbb{R}^4$, $\varphi(\sigma)$ 'close' to $\operatorname{sat}(\sigma)$ $\big(d/d\sigma \not> 0\big)$
- **Leonov G., Kuznetsov N., Bragin V. 2010**:
  analytical-numerical counterexamples construction for any type of nonl.; counterexample in $\mathbb{R}^4$ with $\varphi(\sigma) = \tanh(\sigma)$: $\ 0 < \tanh'(\sigma) \leqslant 1$

Survey: Bragin, Vagaitsev, Kuznetsov, Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544*

Hidden oscillation localization: analytical-numerical procedure

Describing function method (DFM) can lead to untrue results: no periodic solution for Aizerman's or Kalman's conditions by DFM

$(1)\ \dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \varphi(\mathbf{x}) \qquad (2)\ \dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \varepsilon\boldsymbol{\varphi}(\mathbf{x}) \qquad (3)\ \dot{\mathbf{x}}^j = \mathbf{P}_0\mathbf{x}^j + \varepsilon_j\boldsymbol{\varphi}(\mathbf{x}^j)$

ε allows one to justify math. strictly DFM for (2) & to determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$ — *oscillating attractor* $\mathcal{A}_0$.

Localization of attractor $\mathcal{A}$ in (1): numerically follow transformation of $\mathcal{A}_j$ with increasing $j = 0, .., m$ ($\mathcal{A}_m = \mathcal{A}$). Two cases are possible:

1. if all points of $\mathcal{A}_0$ are in the attraction domain of $\mathcal{A}_1$ (oscillating attractor of (3) with $j = 1$), then solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of (3) with $j = 1$ from initial point $\mathbf{x}^0(0)$. If in computational process $\mathbf{x}^1(t)$ is not fallen to equilibria and is not $\to \infty$ (on suff. large $[0, T]$), then $\mathbf{x}^1(t)$ computes attractor $\mathcal{A}_1$. Then perform similar procedure for (3) with $j = 2$: by starting trajectory $\mathbf{x}^2(t)$ of (3) with $j = 2$ from init. point $\mathbf{x}^1(T)$ (last point on previous step) we compute $\mathcal{A}_2$. And so on.

2. in the change from system (2) to (3) with $j = 1$, it's observed loss of stability bifurcation and vanishing of attractor $\mathcal{A}_0$(or $\mathcal{A}_{j-1}$ on j-th step).

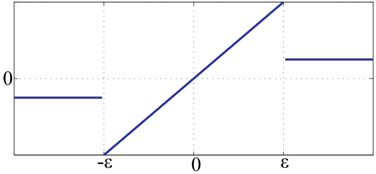
DFM justification for critical case

$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\varphi_\varepsilon(r^*\mathbf{x}),\ \ \mathbf{x} \in \mathbb{R}^n$
eigs($\mathbf{P}$): $\lambda_{1,2} = \pm i\omega_0$, Re $\lambda_{j>2} < 0$
$\Rightarrow \exists \mathbf{x} = Sy$:

$$\dot{y}_1 = -\omega_0 y_1 + b_1\varphi_\varepsilon(y_1 + \mathbf{c}_3^*\mathbf{y}_3)$$
$$\dot{y}_2 = +\omega_0 y_1 + b_2\varphi_\varepsilon(y_1 + \mathbf{c}_3^*\mathbf{y}_3)$$
$$\dot{\mathbf{y}}_3 = \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}\varphi_\varepsilon(y_1 + \mathbf{c}_3^*\mathbf{y}_3)$$

$\mathbf{A}_3$–stable $(n-2)\times(n-2)$-matrix
$\mathbf{b}, \mathbf{c}$ — $(n-2)$-vectors.

$$\varphi_\varepsilon(\sigma)=\begin{cases}\mu\sigma, & \forall|\sigma|\le\varepsilon\\ \mathrm{sign}(\sigma)M\varepsilon^3, & \forall|\sigma|>\varepsilon\end{cases}$$



Stab. sector: $\mu\sigma \geqslant \varphi_\varepsilon \geqslant 0$

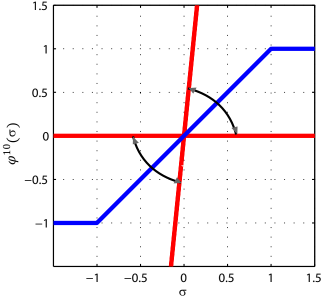
Theorem. If $b_1 < 0$ and $0 < (\mu b_2\omega_0(\mathbf{c}_3^*\mathbf{b}_3 + b_1) + b_1\omega_0^2)$
then for suff. small ε exists a stable periodic solution with the initial data

$$y_1(t)=-\sin(\omega_0 t)y_2(0)+O(\varepsilon),\ y_2(t)=\cos(\omega_0 t)y_2(0)+O(\varepsilon),\ \mathbf{y}_3(t)=O(\varepsilon)$$

$$y_1(0)=O(\varepsilon^2),\ y_2(0)=-\sqrt{\frac{\mu(\mu b_2\omega_0(\mathbf{c}_3^*\mathbf{b}_3+b_1)+b_1\omega_0^2)}{-3\omega_0^2 M b_1}}+O(\varepsilon),\ \mathbf{y}_3(0)=O(\varepsilon^2)$$

Counterexample to Aizerman and Kalman conjecture

$$\dot{x}_1 = -x_2 - 10\,\varphi(\sigma)$$
$$\dot{x}_2 = x_1 - 10.1\,\varphi(\sigma)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 + \varphi(\sigma)$$
$$\sigma = x_1 - 10.1\,x_3 - 0.1\,x_4$$



Thm: $\varphi(\sigma) = \varphi^0(\sigma)$ $\exists$ periodic solution with $x_1(0) = x_3(0) = x_4(0) = 0, x_2(0) = -1.7513$

Aizerman's conjecture: $0 \leqslant \varphi^j(\sigma) \leqslant 1$,

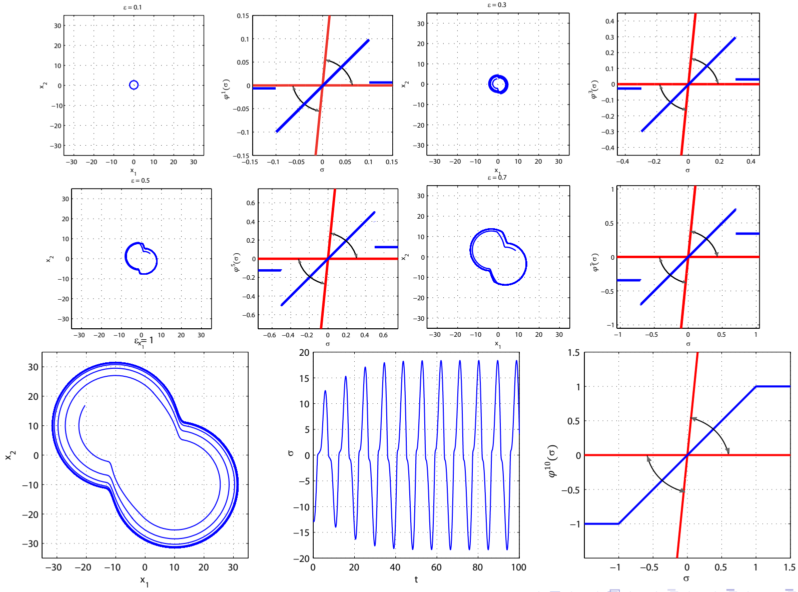
$$\varphi^j(\sigma) = \begin{cases} \sigma, & |\sigma| \le \varepsilon_j; \\ \mathrm{sign}(\sigma)\varepsilon_j^3, & |\sigma| > \varepsilon_j \end{cases} \quad \varepsilon_j = 0.1, ..., 1, \quad \varphi^{10}(\sigma) = \mathrm{sat}(\sigma)$$

Kalman's conjecture: $iN \leqslant {\psi^i}'(\sigma) \leqslant 1 \qquad 0 < \frac{d}{d\sigma}\tanh(\sigma) \le 1$

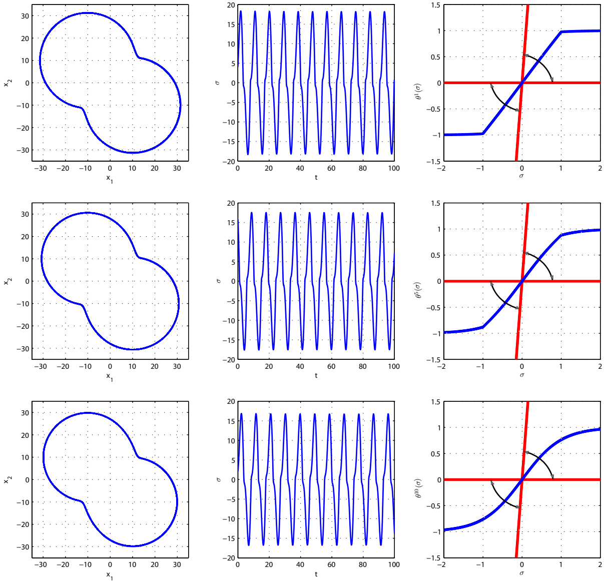
$$\psi^i(\sigma) = \begin{cases} \sigma, & |\sigma| \le 1; \\ \mathrm{sign}(\sigma) + i(\sigma - \mathrm{sign}(\sigma))N, & |\sigma| > 1 \end{cases} \quad N = 0.01, i = 1, .., 5$$

$$\theta^i(\sigma) = \mathrm{sat}(\sigma) + i(\tanh(\sigma) - \mathrm{sat}(\sigma))/10 \quad i = 1, .., 10 \quad \theta^{10}(\sigma) = \tanh(\sigma)$$

Counterexample to Aizerman conjecture



Smooth counterexample to Kalman conjecture



$$\dot{x}_1 = -x_2 - 10\,\varphi(\sigma)$$
$$\dot{x}_2 = x_1 - 10.1\,\varphi(\sigma)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 + \varphi(\sigma)$$
$$\sigma = x_1 - 10.1\,x_3 - 0.1\,x_4$$

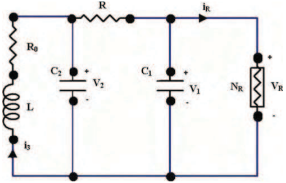
$$\varphi(\sigma) = \theta^i(\sigma) = \mathrm{sat}(\sigma) + i(\tanh(\sigma) - \mathrm{sat}(\sigma))/10$$
$$i = 1, .., 10$$
$$\tanh(\sigma) = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}}$$

Counterexample to Kalman problem (i=10) $(0 < \frac{d}{d\sigma}\tanh(\sigma) \leq 1)$ periodic solution exists, linear systems are stable

Attractors in Chua's circuits

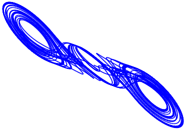
L.Chua (1983)

$$\dot{x} = \alpha(y - x - f(x)),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z),$$



$$f(x) = m_1 x + \mathrm{sat}(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| + |x-1|)$$

Chua circuit can be used in chaotic communications



1983–now: computations of Chua self-excited attractors by standard procedure: trajectory from a neighborhood of unstable equilibrium reaches and identifies attractor. [Bilotta&Pantano, A gallery of Chua attractors, WorldSci. 2008]

Could an attractor exists and how lo localize it, if equilibrium is stable?

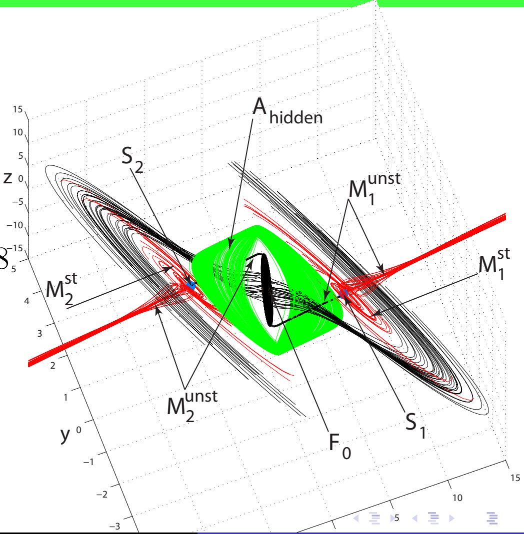
L.Chua, 1992: If zero equilibrium is stable ⇒ there is no attractor

Hidden attractor in classical Chua's system

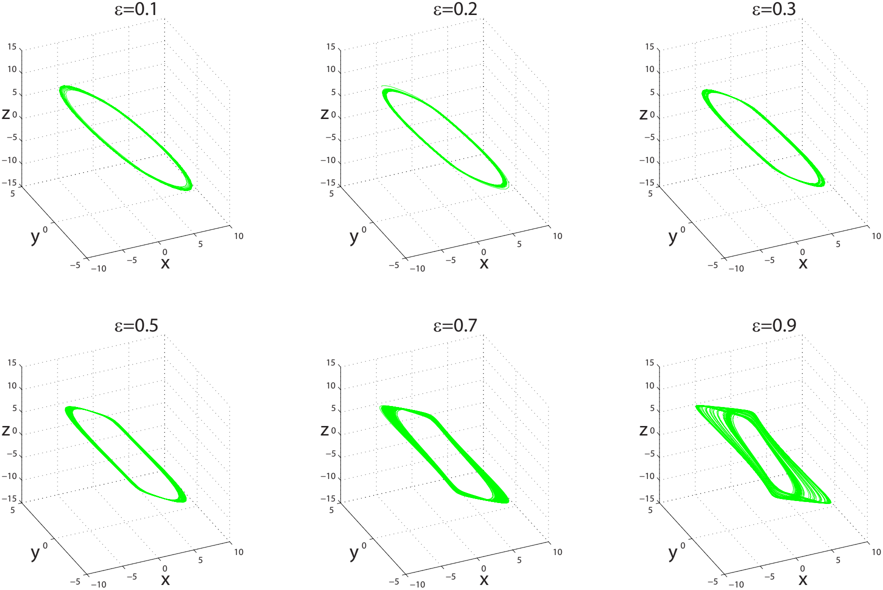
In 2010 the notion of *hidden attractor* was introduced
and hidden chaotic attractor was found for the first time by the authors

$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z,\ \dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$
$$\alpha = 8.4562,\ \beta = 12.0732$$
$$\gamma = 0.0052$$
$$m_0 = -0.1768,\ m_1 = -1.1468$$

Stable zero eqv. and
2 symmetric saddles:
trajectories "from"
saddles tend to
zero eqv. or to infinity:
black and red
Hidden attractor (green)



Transition to chaos scenario



References: 2012

✓ Kuznetsov N., Kuznetsova O., Leonov G., Vagaitsev V., Analytical-numerical localization of hidden attractor in electrical Chua's circuit, **Lecture Notes in Electrical Engineering**, Volume 174 LNEE, **2013**, 149-158 (doi: 10.1007/978-3-642-31353-0_11)
✓ Leonov G.A., Kuznetsov N.V., Vagaitsev V.I., Hidden attractors in smooth Chua's systems, **Physica D**, **2012** (doi: 10.1016/j.physd.2012.05.016)
✓ Kuznetsov N.V., Kuznetsova O.A., Leonov G.A., Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, **Differential equations and Dynamical systems**, **2012** (doi: 10.1007/s12591-012-0118-6)
✓ Leonov G.A., Kuznetsov N.V., Hidden oscillations in dynamical systems: 16 Hilbert's problem, Aizerman's and Kalman's conjectures, hidden attractors in Chua's circuits, **J. Math. Sci.**, **2012**
✓ G.A. Leonov, N.V. Kuznetsov, Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman and Kalman problems to hidden chaotic attractors in Chua circuits, **Int. J. of Bifurcation and Chaos**, **2012** [accepted]
✓ G.A. Leonov, B.R. Andrievsky, A.Yu. Pogromsky, Control of aircrafts with AW-compensation, **Differential equations**, N13, **2012** [in print]
✓ Leonov G.A., Kuznetsov N.V., Yuldahsev M.V., Yuldashev R.V., Analytical method for computation of phase-detector characteristic, **IEEE Transactions on Circuits and Systems Part II**, 59(10), **2012** (doi: 10.1109/TCSII.2012.2213362)
✓ Leonov G.A., Kiseleva M.A., Kuznetsov N.V., Neittaanmaki P., Hidden Oscillations in Drilling systems: Torsional Vibrations, **Journal of Applied Nonlinear Dynamics**, **2012** [accepted]
✓ Kiseleva M.A., Kuznetsov N.V., Leonov G.A., Neittaanmaki P., Drilling Systems Failures and Hidden Oscillations, NSC 2012 - 4th IEEE Int. Conf. on Nonlinear Science and Complexity, **2012**, 109-112 (doi:10.1109/NSC.2012.6304736)

References: 2011

✓ V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman Conjectures and Chua's Circuits, **J. of Computer and Systems Sciences Int.**, **2011**, 50(4), 511-544 (survey) (doi:10.1134/S106423071104006X)
✓ Leonov G.A., Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors , **Physics Letters A**, 375(23), **2011**, 2230-2233 (doi:10.1016/j.physleta.2011.04.037)
✓ G.A. Leonov, N.V. Kuznetsov, O.A. Kuznetsova, S.M. Seledzhi, V.I.Vagaitsev, Hidden oscillations in dynamical systems, *Transaction on Systems and Control*, 6(2), **2011**, 54–67 (survey)
✓ G.A. Leonov, N.V. Kuznetsov, Algorithms for searching for hidden oscillations in the Aizerman and Kalman problems, **Doklady Mathematics**, **2011**, 84(1), 475-481 (doi:10.1134/S1064562411040120)
✓ G.A. Leonov, N.V. Kuznetsov, and E.V. Kudryashova, A Direct Method for Calculating Lyapunov Quantities of Two-Dimensional Dynamical Systems, **Proceedings of the Steklov Institute of Mathematics**, 272(Suppl. 1), **2011**, 119-127 (doi:10.1134/S008154381102009X)
✓ Leonov G.A., Four normal size limit cycle in two-dimensional quadratic system, **Int. J. of Bifurcation and Chaos**, 21(2) **2011**, 425-429
✓ Leonov G.A., Kuznetsov N.V., Analytical-numerical methods for investigation of hidden oscillations in nonlinear control systems, **IFAC Proceedings Volumes (IFAC-PapersOnline)**, 18(1), **2011**, 2494-2505 (doi:10.3182/20110828-6-IT-1002.03315) (survey paper)
✓ Kuznetsov N.V., Leonov G.A., Seledzhi S.M., Hidden oscillations in nonlinear control systems, **IFAC Proceedings Volumes (IFAC-PapersOnline)**, 18(1), **2011**, 2506-2510 (doi:10.3182/20110828-6-IT-1002.03316)
✓ Kuznetsov N.V., Kuznetsova O.A., Leonov G.A., Vagaitsev V.I., Hidden attractor in Chua's circuits, ICINCO 2011 - Proc. of the 8th Int. Conf. on Informatics in Control, Automation and Robotics, Vol. 1, **2011**, 279-282 (doi: 10.5220/0003530702790283)

References: 2010-...

✓ G.A. Leonov, V.O. Bragin, N.V. Kuznetsov, Algorithm for constructing counterexamples to the Kalman problem. **Doklady Mathematics**, 82(1), **2010**, 540-542 (doi:10.1134/S1064562410040101)
✓ G. A. Leonov, Four limit cycles in quadratic two-dimensional systems with a perturbed first order weak focus, **Doklady Mathematics**, **2010**, 81(2), 248-250 (doi:10.1134/S1064562410020237).
✓ G.A. Leonov, N.V. Kuznetsov, Limit cycles of quadratic systems with a perturbed weak focus of order 3 and a saddle equilibrium at infinity, **Doklady Mathematics**, **2010**, 82(2), 693-696 (doi:10.1134/S1064562410050042)
✓ G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov, Algorithm for localizing Chua attractors based on the harmonic linearization method, **Doklady Mathematics**, 82(1), **2010**, 663-666 (doi:10.1134/S1064562410040411)
✓ N.V. Kuznetsov, G.A. Leonov and V.I. Vagaitsev, Analytical-numerical method for attractor localization of generalized Chua's system, **IFAC Proceedings Volumes (IFAC-PapersOnline)**, 4(1), **2010** (doi:10.3182/20100826-3-TR-4016.00009)
✓ Leonov G.A., Kuznetsova O.A., Lyapunov quantities and limit cycles of two-dimensional dynamical systems. Analytical methods and symbolic computation, **R&C dynamics**, 15(2-3), **2010**, 354-377
✓ Leonov G.A., Efficient methods for search of periodical oscillations in dynamic systems. **Applied mathematics and mechanics**, 1, **2010**, 37-73
✓ Leonov G.A., Kuznetsov N.V., Bragin V.O., On Problems of Aizerman and Kalman, **Vestnik St. Petersburg University. Mathematics**, 43(3), **2010**, 148-162 (doi:10.3103/S1063454110030052)
✓ Leonov G.A., Kuznetsov N.V., Kudryashova E.V., Cycles of Two-Dimensional Systems: Computer Calculations, Proofs, and Experiments, **Vestnik St. Petersburg University. Mathematics**, 41(3), **2008**, 216-250 (doi:10.3103/S1063454108030047)
✓ Kuznetsov N.V., Leonov G.A., Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems, **J. of Vibroengineering**, 10(4), **2008**, 460-467

Questions and remarks